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# PGDUS-powered inverse Rayleigh Distribution: Properties and Stress-Strength Reliability Analysis using Maximum Likelihood and Maximum Product Spacing Methods

Amrutha M[a] and V M Chacko[a]

[a] *Department of Statistics, St. Thomas College (Autonomous), Thrissur, University of Calicut, Kerala, India*

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**ABSTRACT**
The Power Generalized DUS transformation provides improved flexibility when dealing with lifetime and reliability data. It provides parsimonious model and is a powerful tool for statistical modeling and analysis of a parallel system or maximum random variable. The empirical success and adaptability of the PGDUS transformation further highlight its value in diverse applications. The Power Generalized DUS powered inverse Rayleigh distribution (PGDUS-PIR) is a new distribution that we introduced in this paper and that was derived by using the PGDUS transformation on a baseline distribution, the powered inverse Rayleigh (PIR) distribution. Its statistical features are discussed in detail. The unknown parameters are estimated using the maximum likelihood method and the maximum product spacing method. To better understand the behavior and applicability of the PGDUS-PIR distribution, a simulation study was carried out, which led to more accurate and reliable statistical modeling and analysis. Two sets of real data are used to compare the performance of the proposed distribution with some distributions that are currently in use. Stress-strength reliability is an essential aspect of lifetime data analysis, providing a systematic approach to evaluate the reliability and durability of components and systems under varying stresses. This concept is essential to ensure the safety, quality and durability of products in many industries. We derived the single- and multicomponent stress-strength reliability of the PGDUS-PIR $(\alpha, \beta, \theta)$ distribution.

**KEYWORDS**
Powered inverse Rayleigh distribution; PGDUS transformation; Maximum likelihood estimation; Maximum Product Spacing estimation; Stress-Strength Reliability

## 1. Introduction

The Rayleigh distribution was first presented by [11] in connection with an issue in the fields of optics and acoustics. The inverse Rayleigh (IR) distribution proposed by [14] is a continuous distribution obtained from the Rayleigh distribution by taking the reciprocal of a Rayleigh random variable. It is commonly used in reliability engineering and survival analysis because of its ability to model the lifetimes and failure times of

CONTACT V M Chacko[a]. Email:chackovm@gmail.com

systems and components. According to the transformation, the IR variable is given as

$$X = 1/Y,$$

where $Y$ is a Rayleigh random variable. Then the probability density function (pdf) of the IR distribution, with a scale parameter $\beta$, is given by

$$f_X(x) = \frac{2}{\beta}\frac{e^{\frac{-1}{\beta x^2}}}{x^3} \quad ; \; x > 0 \;, \beta > 0. \tag{1}$$

The IR distribution is used in reliability engineering to represent component failure time, particularly when failures occur due to stresses that are inversely proportional to a primary variable. Whenever inverse connections among variables are seen in medical and biological studies, the IR distribution is appropriate for modeling lifetimes in survival analysis. IR distribution is used in environmental research to examine data that show heavy-tailed features, including pollutant concentrations or extreme weather events. Numerous characteristics of the IR distribution have been examined by [15]. Through the power transformation, which [4] initially presented, [1] introduced the powered inverse Rayleigh (PIR) distribution. The PIR distribution offers a versatile tool for modeling lifetime and reliability data with decreasing hazard rates and varying tail behaviors. Its applications in reliability engineering, survival analysis, and environmental studies demonstrate its utility in capturing real-world phenomena. The flexibility introduced by the shape parameter $\alpha$ allows for more accurate modeling and analysis, making it a valuable addition to the family of lifetime distributions. If $X$ follows an IR distribution with scale parameter $\beta$, then the random variable $Z = X^{\alpha}$ follows a PIR distribution with shape parameter $\alpha$ and scale parameter $\beta$. Then, the pdf and cumulative distribution function (cdf) of the PIR distribution are defined as

$$f_Z(z) = \frac{2\alpha}{\beta}\frac{e^{-\frac{1}{\beta z^{2\alpha}}}}{z^{2\alpha+1}} \quad ; \qquad z > 0,\; \alpha > 0,\; \beta > 0, \tag{2}$$

$$F_Z(z) = e^{-1/\beta z^{2\alpha}} \quad ; \qquad z > 0,\; \alpha > 0,\; \beta > 0. \tag{3}$$

The DUS transformed PIR distribution (see [7]) is a novel and flexible distribution that combines the characteristics of the PIR distribution with the DUS transformation. The DUS transformation proposed by [9], without adding more parameters, allows for even greater adaptability in modeling various types of data, particularly in fields such as reliability engineering and survival analysis. The DUS transformation of a baseline distribution with pdf $f(b)$ and cdf $F(b)$ can be defined as

$$g(b) = \frac{f(b)e^{F(b)}}{e-1},$$

$$G(b) = \frac{e^{F(b)}-1}{e-1}.$$

It is possible to model real life data using DUS-PIR distribution more effectively.

Hence, pdf and cdf of DUS-PIR distribution are

$$g(z) = \frac{2\alpha}{\beta(e-1)} \frac{e^{-1/\beta z^{2\alpha}} e^{e^{-1/\beta z^{2\alpha}}}}{z^{2\alpha+1}}, \qquad z > 0,\ \alpha > 0,\ \beta > 0, \tag{4}$$

and

$$G(z) = \frac{e^{e^{-1/\beta z^{2\alpha}}} - 1}{e-1}, \qquad z > 0,\ \alpha > 0,\ \beta > 0 \tag{5}$$

respectively.

The power generalized DUS (PGDUS) transformation provides greater flexibility in modeling various types of data (see [13]). This allows the transformed distribution to capture a wider range of behaviors and patterns that might not be possible with the baseline distribution alone. By applying the PGDUS transformation to an existing distribution, the resulting distribution often shows a better fit to empirical data. PGDUS transformation can be used to model a parallel system in which each of the components in the system is distributed as any DUS-transformed baseline distribution. This transformation is nothing but the exponentiation of DUS transformation, and the pdf and cdf of the PGDUS transformed distribution are the following:

$$h(c) = \frac{\theta}{(e-1)^{\theta}} (e^{G(c)} - 1)^{\theta-1} e^{G(c)} g(c),$$
$$H(c) = \left( \frac{e^{G(c)} - 1}{e-1} \right)^{\theta}, \qquad c > 0,\ \theta > 0.$$

where $C$ be a random variable with pdf $g(c)$ and cdf $G(c)$ respectively. In [13], the characteristics of PGDUS exponential distribution are described. [12] studied PGDUS transformation of the Weibull and Lomax distributions to model the parallel systems with independent components, in which the lifetimes of each of the components follow the Weibull and Lomax distributions, respectively.

The stress-strength reliability can be mathematically represented as $R = Pr(Y < X)$, which evaluates the probability that the random strength of a component $(X)$ will be greater than its stress $(Y)$ (see [8]). It would be useful to study the distributional properties, applications and stress-strength analysis of the PGDUS transformation with the PIR distribution as a baseline model, since many of the lifetime data fits with the PIR distribution.

The paper is structured as follows. In Section 2, we suggested a new lifetime distribution by making use of the PGDUS transformation with the PIR distribution as a baseline model and then described its statistical properties in Section 3. Section 4 discussed the various estimate techniques, including the maximum likelihood and maximum product spacing techniques, for the proposed distribution. In Section 5, a simulation study is conducted to evaluate the estimators' performance. In Section 6, a set of successive precipitation data in March for Minneapolis/St.Paul and the daily cases of COVID-19 is provided to examine the model. Section 7 discussed the theory of stress-strength reliability in both single- and multicomponent models. Also, computation of the reliability estimate for single- and multicomponent stress-strength models is carried out. In this section, a simulation study is also included to evaluate how well the reliability estimations perform. In Section 8, a computation of $Pr(Y < X)$, is given for real-life data on the GPA strength of carbon fibers at two different lengths.

In Section 9, conclusions are given.

## 2. Power Generalized DUS powered inverse Rayleigh Distribution

Assume that $X$ is a random variable with a PIR distribution and that (2) and (3) provide its pdf and cdf, respectively. So the pdf, cdf, and the failure rate function of PGDUS transformed distribution, which is called the PGDUS-PIR with parameters $\alpha, \beta$ and $\theta$, can be defined as

$$h(x) = \frac{2\alpha\theta}{\beta(e-1)^{\theta}} \frac{e^{-1/\beta x^{2\alpha}}}{x^{2\alpha+1}} e^{e^{-1/\beta x^{2\alpha}}} \left(e^{e^{-1/\beta x^{2\alpha}}} - 1\right)^{\theta-1}, \quad x > 0,\ \alpha > 0,\ \beta > 0,\ \theta > 0, \tag{6}$$

$$H(x) = \left(\frac{e^{e^{-1/\beta x^{2\alpha}}} - 1}{e-1}\right)^{\theta}, \quad x > 0,\ \alpha > 0,\ \beta > 0,\ \theta > 0, \tag{7}$$

and

$$F_R(x) = \frac{2\alpha e^{-1/\beta x^{2\alpha}} e^{e^{-1/\beta x^{2\alpha}}} \left(e^{e^{-1/\beta x^{2\alpha}}} - 1\right)^{\theta-1}}{\beta x^{2\alpha+1}\left((e-1)^{\theta} - \left(e^{e^{-1/\beta x^{2\alpha}}} - 1\right)^{\theta}\right)}, \quad x > 0,\ \alpha > 0,\ \beta > 0,\ \theta > 0 \tag{8}$$

respectively.

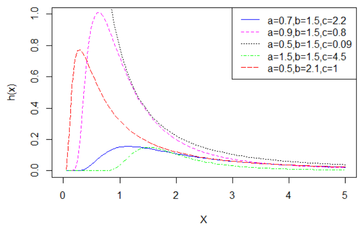


**Figure 1.** pdf plot for PGDUS-PIR distribution

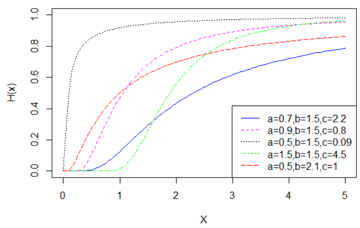


**Figure 2.** cdf plot for PGDUS-PIR distribution

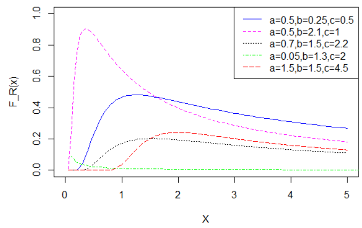


**Figure 3.** Failure rate plot of PGDUS-PIR distribution

With varying values for each parameter, Figure 1, 2, and 3 show the graphical representations of the pdf, cdf, and failure rate function of the PGDUS-PIR distribution, respectively. The failure rate plot of the PGDUS-PIR$(\alpha, \beta, \theta)$ distribution has an upside-down bathtub shape.

## 3. Statistical Properties

Statistical characteristics such as moments, order statistics, quantile function, and entropy of the PGDUS-PIR $(\alpha, \beta, \theta)$ can be obtained as follows:

### 3.1. Moments

The $a^{th}$ order raw moment is described as

$$\begin{aligned}\mu_a^1 &= E(X^a)\\ &= \int_0^\infty x^a \frac{2\alpha\theta}{\beta(e-1)^\theta} \frac{e^{-1/\beta x^{2\alpha}}}{x^{2\alpha+1}} e^{e^{-1/\beta x^{2\alpha}}} \left(e^{e^{-1/\beta x^{2\alpha}}} - 1\right)^{\theta-1} dx \end{aligned} \tag{9}$$

Substitute $w = e^{-1/\beta x^{2\alpha}}$, $\quad dw = \frac{2\alpha}{\beta} \frac{e^{-1/\beta x^{2\alpha}}}{x^{2\alpha+1}} dx$

$$\begin{aligned}\mu_a^1 &= \frac{\theta}{(e-1)^\theta} \int_0^\infty (-\beta \log w)^{\frac{-a}{2\alpha}} e^w (e^w - 1)^{\theta-1} dw\\ &= \frac{\theta}{(e-1)^\theta} \sum_{k=0}^{\theta-1} (-1)^{k-\frac{a}{2\alpha}} \beta^{\frac{a}{2\alpha}} \binom{\theta-1}{k} \int_0^\infty (\log(u))^{\frac{-a}{2\alpha}} e^{u(\theta-k)} du\\ &= \frac{\theta}{(e-1)^\theta} \sum_{m=0}^{\infty} \sum_{k=0}^{\theta-1} (-1)^{k-\frac{a}{2\alpha}} \beta^{\frac{a}{2\alpha}} \binom{\theta-1}{k} \frac{(\theta-k)^m}{m!} \int_0^1 u^m (\log(u))^{\frac{-a}{2\alpha}} du\\ &= \frac{\theta}{(e-1)^\theta} \sum_{m=0}^{\infty} \sum_{k=0}^{\theta-1} (-1)^{k-\frac{a}{2\alpha}} \beta^{\frac{a}{2\alpha}} \binom{\theta-1}{k} \frac{(\theta-k)^m}{m!} (1+m)^{\frac{a}{2\alpha}-1} e^{\frac{-i\pi a}{2\alpha}} \Gamma\big(1 - \frac{a}{2\alpha}\big),\end{aligned} \tag{10}$$

where $Re(\frac{a}{\alpha}) < 2$ and $Re(m) > -1$.
By substituting the values $a = 1$ and $a = 2$, we can derive the corresponding mean

and distribution's variance. That is,

$$\mu_1^1 = \frac{\theta}{(e-1)^\theta}\sum_{m=0}^{\infty}\sum_{k=0}^{\theta-1}(-1)^{k-\frac{1}{2\alpha}}\beta^{\frac{1}{2\alpha}}\binom{\theta-1}{k}\frac{(\theta-k)^m}{m!}\big(1+m\big)^{\frac{1}{2\alpha}-1}e^{\frac{-i\pi}{2\alpha}}\Gamma\big(1-\frac{1}{2\alpha}\big), \tag{11}$$

$$Re(\frac{1}{\alpha}) < 2,\ Re(m) > -1.$$

$$\mu_2^1 = \frac{\theta}{(e-1)^\theta}\sum_{m=0}^{\infty}\sum_{k=0}^{\theta-1}(-1)^{k-\frac{1}{\alpha}}\beta^{\frac{1}{\alpha}}\binom{\theta-1}{k}\frac{(\theta-k)^m}{m!}\big(1+m\big)^{\frac{1}{\alpha}-1}e^{\frac{-i\pi}{\alpha}}\Gamma\big(1-\frac{1}{\alpha}\big), \tag{12}$$

$$Re(\frac{2}{\alpha}) < 2,\ Re(m) > -1.$$

Then,
Variance $= \mu_2^1 - \big(\mu_1^1\big)^2$.

### *3.2. Quantile Function*

Let $D(i)$ denote the $i^{th}$ quantile of the PGDUS-PIR $(\alpha, \beta, \theta)$ distribution. $i^{th}$ quantile function can be obtained by solving the following condition

$$H(D(i)) = i, \qquad 0 < i < 1.$$

That is,

$$\left(\frac{e^{e^{\frac{-1}{\beta D^{2\alpha}}}}-1}{e-1}\right)^\theta = i$$

Therefor,

$$D(i) = \bigg(-\beta\log\big(\log\big(1+i^{1/\theta}(e-1)\big)\big)\bigg)^{\frac{-1}{2\alpha}}, \qquad i\ \epsilon\ (0,1),\ \alpha>0,\ \beta>0,\ \theta>0. \tag{13}$$

### *3.3. Order Statistics*

Order statistics are fundamental in statistical theory and practice, representing the statistics obtained from the ordered values of a sample. Given a sample of size $n$ from the population PGDUS-PIR$(\alpha, \beta, \theta)$, then the pdf and cdf of $r^{th}$ order statistics, $X_{(r)}$, of the proposed distribution can be defined as follows:

$$
\begin{aligned}
h_{(r)}(x) &= \frac{n!}{(r-1)!(n-r)!} h(x)(H(x))^{r-1}\big(1-H(x)\big)^{n-r} \\
&= \frac{2\alpha\theta(n!)e^{\frac{-1}{\beta x^{2\alpha}}}}{(r-1)!(n-r)!} \frac{e^{e^{\frac{-1}{\beta x^{2\alpha}}}}\big(e^{e^{\frac{-1}{\beta x^{2\alpha}}}}-1\big)^{\theta-1}}{\beta x^{2\alpha+1}(e-1)^{\theta}} \left(\frac{e^{e^{\frac{-1}{\beta x^{2\alpha}}}}-1}{e-1}\right)^{\theta(r-1)} \left(1-\left(\frac{e^{e^{\frac{-1}{\beta x^{2\alpha}}}}-1}{e-1}\right)^{\theta}\right)^{n-r} \\
&= \frac{(n!)2\alpha\theta}{(r-1)!(n-r)!} \frac{e^{\frac{-1}{\beta x^{2\alpha}}} e^{e^{\frac{-1}{\beta x^{2\alpha}}}}}{\beta(e-1)^{n\theta}} \frac{\big(e^{e^{\frac{-1}{\beta x^{2\alpha}}}}-1\big)^{\theta r-1}}{x^{2\alpha+1}} \left((e-1)^{\theta}-\big(e^{e^{\frac{-1}{\beta x^{2\alpha}}}}-1\big)^{\theta}\right)^{n-r},
\end{aligned} \tag{14}
$$

$$
\begin{aligned}
F_{(r)}(x) &= \sum_{s=r}^{n} \binom{n}{s} (F(x))^{s}(1-F(x))^{n-s} \\
&= \sum_{s=r}^{n} \binom{n}{s} \frac{\big(e^{e^{\frac{-1}{\beta x^{2\alpha}}}}-1\big)^{\theta s}}{(e-1)^{n\theta}} \left((e-1)^{\theta}-\big(e^{e^{\frac{-1}{\beta x^{2\alpha}}}}-1\big)^{\theta}\right)^{n-r},
\end{aligned} \tag{15}
$$

where $x>0,\ \ \alpha>0,\ \beta>0$, and $\theta>0$.

### *3.4. Entropy*

Entropy of lifetime distributions measures the uncertainty or randomness associated with the time until an event of interest occurs. This concept is particularly useful in reliability engineering and survival analysis, where understanding the variability and predictability of lifetimes is crucial. Shannon entropy quantifies the uncertainty associated with the lifetime of a component or system.

Renyi entropy is a generalization of the Shannon entropy, which provides a parameterized family of entropy measures that can be used to assess the uncertainty of lifetime distributions. For a continuous random variable $X$ representing the lifetime with pdf $h(x)$, the Renyi entropy of order $\Lambda$, where $\Lambda>0$ and $\Lambda\neq 1$, is defined as:

$$\tau_R(\Lambda) = \frac{1}{1-\Lambda}\log\left(\int h^{\Lambda}(x)dx\right)$$

$$
\begin{aligned}
\int_0^{\infty} h^{\Lambda}(x)dx &= \left(\frac{2\alpha\theta}{\beta(e-1)^{\theta}}\right)^{\Lambda} \int_0^{\infty} \frac{e^{\frac{-\Lambda}{\beta x^{2\alpha}}}}{x^{2\alpha+1}} e^{\Lambda e^{\frac{-1}{\beta x^{2\alpha}}}} \big(e^{\Lambda e^{\frac{-1}{\beta x^{2\alpha}}}}-1\big)^{\Lambda(\theta-1)}dx \\
&= \left(\frac{2\alpha\theta}{\beta(e-1)^{\theta}}\right)^{\Lambda} \sum_{k=0}^{\Lambda(\theta-1)} (-1)^k \binom{\Lambda(\theta-1)}{k} \int_0^{\infty} \frac{e^{\frac{-\Lambda}{\beta x^{2\alpha}}}}{x^{2\alpha+1}} e^{(\Lambda(\theta-1)-k+1)\Lambda e^{\frac{-1}{\beta x^{2\alpha}}}} dx \\
&= \left(\frac{2\alpha\theta}{\beta(e-1)^{\theta}}\right)^{\Lambda} \sum_{m=0}^{\infty}\sum_{k=0}^{\Lambda(\theta-1)} (-1)^k \binom{\Lambda(\theta-1)}{k} \frac{\big(\Lambda(\Lambda(\theta-1)-k+1)\big)^m}{m!} \\
&\qquad \int_0^{\infty} \frac{1}{x^{2\alpha+1}} e^{\frac{-(\Lambda+m)}{\beta x^{2\alpha}}} dx \\
&= \left(\frac{2\alpha}{\beta}\right)^{\Lambda-1} \left(\frac{\theta}{(e-1)^{\theta}}\right)^{\Lambda} \sum_{m=0}^{\infty}\sum_{k=0}^{\Lambda(\theta-1)} (-1)^k \binom{\Lambda(\theta-1)}{k} \frac{\big(\Lambda(\Lambda(\theta-1)-k+1)\big)^m}{(m!)(\Lambda+m)}
\end{aligned}
$$

Hence,

$$\tau_R(\Lambda) = \frac{1}{1-\Lambda}\log\left(\left(\frac{2\alpha}{\beta}\right)^{\Lambda-1}\left(\frac{\theta}{(e-1)^{\theta}}\right)^{\Lambda}\sum_{m=0}^{\infty}\sum_{k=0}^{\Lambda(\theta-1)}(-1)^k\binom{\Lambda(\theta-1)}{k}\frac{\left(\Lambda(\Lambda(\theta-1)-k+1)\right)^m}{(m!)(\Lambda+m)}\right), \quad \alpha>0,\ \beta>0,\ \theta>0,\ \Lambda>0 \text{ and } \Lambda\neq 1. \tag{16}$$

## 4. Estimation

Estimation techniques are critical tools in statistical analysis and are used to infer the values of population parameters based on sample data. Here, we examine the method of maximum likelihood estimation (MLE) and the method of maximum product spacing estimation (MPSE). In this section, these techniques are applied to obtain the parameter estimators of the PGDUS-PIR $(\alpha, \beta, \theta)$.

### *4.1. Maximum Likelihood Estimation*

Let $x_1, x_2, ..., x_n$ be a random sample of size $n$ obtained from the PGDUS-PIR$(\alpha, \beta, \theta)$ distribution with pdf given in (6). The likelihood function will take the following form.

$$L(x) = \prod_{i=1}^{n} h(x_i) = \left(\frac{2\alpha\theta}{\beta(e-1)^{\theta}}\right)^n e^{\sum_{i=1}^{n}\frac{-1}{\beta x_i^{2\alpha}}} e^{\sum_{i=1}^{n} e^{-1/\beta x_i^{2\alpha}}} \prod_{i=1}^{n}\frac{\left(e^{e^{-1/\beta x_i^{2\alpha}}}-1\right)^{\theta-1}}{x_i^{2\alpha+1}}. \tag{17}$$

$$\begin{aligned}\log L &= n\log\left(\frac{2\alpha\theta}{\beta(e-1)^{\theta}}\right) - \sum_{i=1}^{n}\frac{1}{\beta x_i^{2\alpha}} + \sum_{i=1}^{n} e^{-1/\beta x_i^{2\alpha}} - (2\alpha+1)\sum_{i=1}^{n}\log(x_i) \\ &\quad + (\theta-1)\sum_{i=1}^{n}\log\left(e^{e^{-1/\beta x_i^{2\alpha}}}-1\right) \\ &= n\log(2\alpha) - n\log(\beta) - n\theta\log(e-1) - (2\alpha+1)\sum_{i=1}^{n}\log(x_i) - \sum_{i=1}^{n}\frac{1}{\beta x_i^{2\alpha}} + \\ &\quad \sum_{i=1}^{n} e^{-1/\beta x_i^{2\alpha}} + (\theta-1)\sum_{i=1}^{n}\log\left(e^{e^{-1/\beta x_i^{2\alpha}}}-1\right).\end{aligned}$$

Solving the following system of equations will yield the MLE of the parameters $\alpha, \beta$ and $\theta$.

$$\frac{\partial \log L}{\partial \alpha} = 0 => \frac{n}{\alpha} + \frac{1}{\beta}\sum_{i=1}^{n} x_i^{-2\alpha}\log(x_i)\left(2e^{-1/\beta x_i^{2\alpha}}+1\right) - 2\sum_{i=1}^{n}\log(x_i) + \frac{2(\theta-1)}{\beta}\sum_{i=1}^{n}\frac{e^{-1/\beta x_i^{2\alpha}} e^{e^{-1/\beta x_i^{2\alpha}}}\log(x_i)}{\left(e^{e^{-1/\beta x_i^{2\alpha}}}-1\right)} = 0. \tag{18}$$

$$\frac{\partial \log L}{\partial \beta} = 0 => \frac{1}{\beta}\Big(\frac{1}{\beta}\sum_{i=1}^{n} x_i^{-2\alpha} - n\Big) + \frac{1}{\beta^2}\sum_{i=1}^{n}\frac{e^{-1/\beta x_i^{2\alpha}}}{x_i^{2\alpha}} + \frac{\theta-1}{\beta^2}\sum_{i=1}^{n}\frac{e^{-1/\beta x_i^{2\alpha}} e^{e^{-1/\beta x_i^{2\alpha}}}}{x_i^{2\alpha}\big(e^{e^{-1/\beta x_i^{2\alpha}}}-1\big)} = 0. \tag{19}$$

$$\frac{\partial \log L}{\partial \theta} = 0 => \frac{n}{\theta} - n\log(e-1) + \sum_{i=1}^{n}\log(e^{e^{-1/\beta x_i^{2\alpha}}}-1). \tag{20}$$

Those systems of equations can be solved using any numerical method to find the estimators.

In order to obtain alternative estimates of the parameters, we consider MPS estimation.

### *4.2. Maximum Product Spacing estimation*

The MPSE method is based on the concept of spacing, which are differences of the CDF, evaluated for a pair of consecutive-order statistics. It aims to maximize the geometric mean of the spacings, there by ensuring a good fit between the model and the observed data. Let $x_{(1)} < x_{(2)} < ... < x_{(n)}$ be an ordered random sample of size $n$ drawn from the PGDUS-PIR$(\alpha, \beta, \theta)$ distribution with cdf $H(x, \underline{\theta})$. Then we can define the spacing of the sample as

$$S_i = H(x_{(i)}) - H(x_{(i-1)}) = \left(\frac{e^{e^{-1/\beta x_{(i)}^{2\alpha}}}-1}{e-1}\right)^{\theta} - \left(\frac{e^{e^{-1/\beta x_{(i-1)}^{2\alpha}}}-1}{e-1}\right)^{\theta}, \quad i = 1, 2, ..., n+1. \tag{21}$$

Let the geometric mean will be

$$A = \left(\prod_{i=1}^{n+1} S_i\right)^{1/n+1} = \left(\prod_{i=1}^{n+1}\left(\frac{e^{e^{-1/\beta x_{(i)}^{2\alpha}}}-1}{e-1}\right)^{\theta} - \left(\frac{e^{e^{-1/\beta x_{(i-1)}^{2\alpha}}}-1}{e-1}\right)^{\theta}\right)^{1/n+1}. \tag{22}$$

$$\log A = \frac{1}{n+1}\sum_{i=1}^{n+1}\log\left(\left(\frac{e^{e^{-1/\beta x_{(i)}^{2\alpha}}}-1}{e-1}\right)^{\theta} - \left(\frac{e^{e^{-1/\beta x_{(i-1)}^{2\alpha}}}-1}{e-1}\right)^{\theta}\right).$$

Consider the first order partial derivatives of $\log A$ with respect to the parameters and equate it with zero as

$$\frac{\partial \log A}{\partial \alpha} = 0 => \frac{2\beta\theta}{n+1}\sum_{i=1}^{n+1}\left(\frac{x_{(i)}^{-2\alpha}\log(x_{(i)})e^{-1/\beta x_{(i)}^{2\alpha}} e^{e^{-1/\beta x_{(i)}^{2\alpha}}}\left(e^{e^{-1/\beta x_{(i)}^{2\alpha}}}-1\right)^{\theta-1}}{\left(e^{e^{-1/\beta x_{(i)}^{2\alpha}}}-1\right)^{\theta} - \left(e^{e^{-1/\beta x_{(i-1)}^{2\alpha}}}-1\right)^{\theta}} - \frac{x_{(i-1)}^{-2\alpha}\log(x_{(i-1)})e^{-1/\beta x_{(i-1)}^{2\alpha}} e^{e^{-1/\beta x_{(i-1)}^{2\alpha}}}\left(e^{e^{-1/\beta x_{(i-1)}^{2\alpha}}}-1\right)^{\theta-1}}{\left(e^{e^{-1/\beta x_{(i)}^{2\alpha}}}-1\right)^{\theta} - \left(e^{e^{-1/\beta x_{(i-1)}^{2\alpha}}}-1\right)^{\theta}}\right) = 0. \tag{23}$$

$$\frac{\partial \log A}{\partial \beta} = 0 => \frac{\theta}{\beta^2(n+1)} \sum_{i=1}^{n+1} \Bigg( \frac{x_{(i)}^{-2\alpha} e^{-1/\beta x_{(i)}^{2\alpha}} e^{e^{-1/\beta x_{(i)}^{2\alpha}}} \left(e^{e^{-1/\beta x_{(i)}^{2\alpha}}} - 1\right)^{\theta-1}}{\left(e^{e^{-1/\beta x_{(i)}^{2\alpha}}} - 1\right)^{\theta} - \left(e^{e^{-1/\beta x_{(i-1)}^{2\alpha}}} - 1\right)^{\theta}} - \frac{x_{(i-1)}^{-2\alpha} e^{-1/\beta x_{(i-1)}^{2\alpha}} e^{e^{-1/\beta x_{(i-1)}^{2\alpha}}} \left(e^{e^{-1/\beta x_{(i-1)}^{2\alpha}}} - 1\right)^{\theta-1}}{\left(e^{e^{-1/\beta x_{(i)}^{2\alpha}}} - 1\right)^{\theta} - \left(e^{e^{-1/\beta x_{(i-1)}^{2\alpha}}} - 1\right)^{\theta}} \Bigg) = 0. \tag{24}$$

$$\frac{\partial \log A}{\partial \theta} = 0 => \frac{1}{n+1} \sum_{i=1}^{n+1} \Bigg( \frac{\left(e^{e^{-1/\beta x_{(i)}^{2\alpha}}} - 1\right)^{\theta} \log\left(\frac{e^{e^{-1/\beta x_{(i)}^{2\alpha}}} - 1}{e-1}\right)}{\left(e^{e^{-1/\beta x_{(i)}^{2\alpha}}} - 1\right)^{\theta} - \left(e^{e^{-1/\beta x_{(i-1)}^{2\alpha}}} - 1\right)^{\theta}} - \frac{\left(e^{e^{-1/\beta x_{(i-1)}^{2\alpha}}} - 1\right)^{\theta} \log\left(\frac{e^{e^{-1/\beta x_{(i-1)}^{2\alpha}}} - 1}{e-1}\right)}{\left(e^{e^{-1/\beta x_{(i)}^{2\alpha}}} - 1\right)^{\theta} - \left(e^{e^{-1/\beta x_{(i-1)}^{2\alpha}}} - 1\right)^{\theta}} \Bigg) = 0. \tag{25}$$

The MPS estimator of $\alpha, \beta$, and $\theta$ that is obtained by solving these systems of equations, respectively.

## 5. Simulation Study

A simulation study is a powerful tool used in statistics to assess the performance of different estimation techniques and statistical methods. It involves generating synthetic data under controlled conditions to evaluate how well various methods perform with regard to bias, mean squared error (MSE), and other relevant metrics. Here, the performance of MLE and MPSE methods is evaluated. It shows that the performance of all methods improved with increasing sample sizes, since the bias and MSE values are getting close to zero.

**Table 1.** Simulation studies for the values $\alpha = 0.5, \beta = 1.5, \theta = 0.09$

| Method | n | bias($\hat{\alpha}$) | bias($\hat{\beta}$) | bias($\hat{\theta}$) | MSE($\hat{\alpha}$) | MSE($\hat{\beta}$) | MSE($\hat{\theta}$) |
|---|---|---|---|---|---|---|---|
| MLE | 25 | -0.12773 | 0.40977 | 0.73808 | 0.03892 | 0.40191 | 1.29703 |
| | 50 | -0.05500 | 0.17526 | 0.31632 | 0.01682 | 0.17111 | 0.55587 |
| | 80 | -0.01916 | 0.06105 | 0.11071 | 0.00583 | 0.05929 | 0.19455 |
| | 100 | -0.01127 | 0.03605 | 0.06502 | 0.00343 | 0.03518 | 0.11426 |
| | 150 | -0.00239 | 0.00759 | 0.01406 | 0.00072 | 0.00721 | 0.02471 |
| | 200 | -0.00029 | 0.00099 | 0.00176 | 8.9716e-05 | 0.00097 | 0.00309 |
| MPS | 25 | 0.11542 | 0.23168 | 0.00262 | 0.01789 | 0.88428 | 0.00438 |
| | 50 | 0.08781 | 0.13029 | 0.00141 | 0.00982 | 0.31912 | 0.00142 |
| | 80 | 0.06699 | 0.086942 | 0.00303 | 0.00557 | 0.13665 | 0.00059 |
| | 100 | 0.06275 | 0.05143 | 0.00144 | 0.00471 | 0.08694 | 0.00035 |
| | 150 | 0.05521 | 0.02026 | 0.00295 | 0.00343 | 0.02544 | 0.00020 |
| | 200 | 0.05200 | 0.01299 | 0.00347 | 0.00299 | 0.02011 | 0.00014 |

Table 1 shows the simulation results of the estimation techniques MLE and MPSE. Based on these results, the performance of the MLE and MPSE techniques for the PGDUS-PIR$(\alpha, \beta, \theta)$ distribution is evaluated across various sample sizes, and it is observed that biases and MSE values in both estimate techniques reduce with increasing sample size.

## 6. Data Analysis

### *Dataset 1*

Here, a set of 30 data of successive values of precipitation (given in Table 2), measured in inches, on March for Minneapolis/St Paul is considered, [6].

**Table 2.** Successive values of March precipitation for Minneapolis/St Paul

| | | | | | |
|---|---|---|---|---|---|
| 0.77 | 1.74 | 0.81 | 1.20 | 1.95 | 1.20 |
| 0.47 | 1.43 | 3.37 | 2.20 | 3.00 | 3.09 |
| 1.51 | 2.10 | 0.52 | 1.62 | 1.31 | 0.32 |
| 0.59 | 0.81 | 2.81 | 1.87 | 1.18 | 1.35 |
| 4.75 | 2.48 | 0.96 | 1.89 | 0.90 | 2.05 |

**Table 3.** Data Analysis

| Distribution | | Estimates | K-S(p-value) | CVM(p-value) | Log(L) | AIC | BIC |
|---|---|---|---|---|---|---|---|
| PGDUS-PIR | MLE | 0.85711<br>1.52230<br>1.13328 | 0.14442<br>(0.5588) | 0.10189<br>(0.5787) | -41.23206 | 86.46411 | 89.26651 |
| | MPS | 0.78924<br>1.63461<br>1.12751 | 0.1624<br>(0.4074) | 0.14461<br>(0.4082) | -41.56992 | 87.13984 | 89.94223 |
| DUS-PIR | MLE | 0.86021<br>1.33230 | 0.14567<br>(0.5476) | 0.10189<br>(0.5691) | -41.23814 | 86.47627 | 89.27867 |
| | MPS | 0.79246<br>1.43909 | 0.16364<br>(0.3978) | 0.14748<br>(0.3991) | -41.57798 | 87.15596 | 89.95836 |
| PIR | MLE | 0.77480<br>0.01238 | 0.15235<br>(0.4893) | 0.12018<br>(0.4971) | -41.91701 | 87.83402 | 90.63642 |
| | MPS | 0.71245<br>1.04413 | 0.1693<br>(0.3562) | 0.16686<br>(0.3434) | -42.26779 | 88.53559 | 91.33798 |

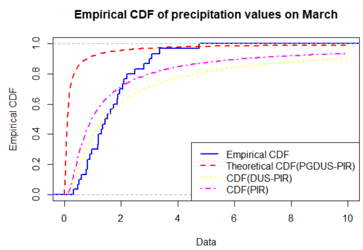


**Figure 4.** Empirical cdf of successive values of precipitation on March for Minneapoolis/St. Paul

The plot of the empirical distribution function (Figure 4) shows that the empirical distribution of the respective data is close to the theoretical cdf of the proposed distribution PGDUS-PIR than the other distributions, DUS-PIR and PIR.

Here, the parameters are estimated by implementing the MLE and MPSE techniques. Goodness of fit is assessed using the Kolmogorov-Smirnov(K-S) test and Cramer-von Mises(CVM) test with p-value, Akaike Information Criteria(AIC), Bayesian Information Criteria(BIC) and log-likelihood values.

The K-S statistic and the CVM statistic for the PGDUS-PIR distribution have the smallest values, with a high p-value. PGDUS-PIR distribution has the smallest AIC and BIC and highest log-likelihood values compared to the other distributions, the DUS-PIR and the PIR distribution, respectively.

Therefore, the data are more appropriate for the PGDUS-PIR distribution than for the DUS-PIR distribution and the PIR distribution.

### *Dataset 2*

The following data (Table 4) represent the daily cases of COVID-19 throughout the world reported by WHO(2020) between January 21 and March 27, 2020 ([10]).

**Table 4.** Daily cases of COVID-19 across the world between January 21 to March 27, 2020

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 60 | 32 | 265 | 472 | 698 | 785 | 1781 | 1477 |
| 1755 | 2010 | 2127 | 2603 | 2838 | 3239 | 3915 | 3721 |
| 3173 | 3437 | 2676 | 3001 | 2546 | 2035 | 14153 | 5151 |
| 2662 | 2097 | 2132 | 2003 | 1852 | 516 | 977 | 996 |
| 978 | 554 | 882 | 741 | 992 | 1292 | 1503 | 1989 |
| 1981 | 1858 | 2573 | 2298 | 3111 | 3625 | 4049 | 3892 |
| 4390 | 4567 | 7266 | 8295 | 11059 | 13042 | 12897 | 15745 |
| 20585 | 26158 | 30648 | 29429 | 32480 | 41371 | 43744 | 10907 |
| 48461 | 60830 | 64501 | | | | | |

**Table 5.** Data Analysis

| Distribution | | Estimates | K-S(p-value) | CVM(p-value) | Log(L) | AIC | BIC |
|---|---|---|---|---|---|---|---|
| PGDUS-PIR | MLE | 0.33303<br>0.02493<br>2.22725 | 0.14272<br>(0.1182) | 0.32399<br>(0.1157) | -669.8411 | 1343.682 | 1348.092 |
| | MPS | 0.32126<br>0.02989<br>2.15048 | 0.15761<br>(0.064) | 0.39826<br>(0.07274) | -670.0571 | 1344.114 | 1348.524 |
| PGDUS-IK | MLE | 0.66612<br>40.06723<br>2.21093 | 0.1449<br>(0.1085) | 0.32648<br>(0.1139) | -670.038 | 1344.076 | 1348.485 |
| | MPS | 0.64333<br>33.65109<br>2.12648 | 0.16041<br>(0.05665) | 0.40375<br>(0.07034) | -670.2468 | 1344.494 | 1348.903 |
| DUS-PIR | MLE | 0.336601<br>0.01031 | 0.15278<br>(0.07863) | 0.37525<br>(0.08381) | -670.6239 | 1345.248 | 1349.657 |
| | MPS | 0.32507<br>0.01280 | 0.16774<br>(0.04072) | 0.46017<br>(0.05004) | -670.8432 | 1345.686 | 1350.096 |
| PIR | MLE | 0.30463<br>0.01160 | 0.1489<br>(0.09233) | 0.38429<br>(0.07926) | -671.8786 | 1347.757 | 1352.166 |
| | MPS | 0.29382<br>0.01418 | 0.16327<br>(0.05090) | 0.46393<br>(0.05093) | -672.0963 | 1348.193 | 1352.602 |

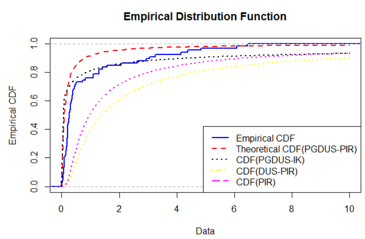


**Figure 5.** Empirical cdf of COVID-19 data and theoretical cdf

From the analysis values (given in Table 5), the PGDUS-PIR $(\alpha, \beta, \theta)$ distribution perfectly fits the COVID-19 data set at a reasonable significance level compared to the other distributions PGDUS-IK, DUS-PIR and PIR, respectively.

The plot of the empirical distribution function (given in Figure 5) shows that the empirical data align more closely with the theoretical cdf of the PGDUS-PIR distribution compared to the PGDUS-IK, DUS-PIR, and PIR distributions.

## 7. Stress-Strength Reliability

### 7.1. Single-component Stress-Strength Reliability

Let $X$ and $Y$ be two independent random variables from PGDUS-PIR distribution with parameters $\alpha, \beta, \theta_1$ and $\theta_2$ respectively. The stress-strength reliability can therefore be described as

$$
\begin{aligned}
Pr(Y < X) &= \int_0^\infty f(x) G_y(x) dx \\
&= \int_0^\infty \frac{2\alpha\theta_1}{\beta(e-1)^{\theta_1}} \frac{e^{-1/\beta x^{2\alpha}}}{x^{2\alpha+1}} e^{e^{-1/\beta x^{2\alpha}}} \left(e^{e^{-1/\beta x^{2\alpha}}} - 1\right)^{\theta_1 - 1} \left(\frac{e^{e^{-1/\beta x^{2\alpha}}} - 1}{e-1}\right)^{\theta_2} dx \\
&= \frac{2\alpha\theta_1}{\beta(e-1)^{\theta_1+\theta_2}} \int_0^\infty \frac{e^{-1/\beta x^{2\alpha}}}{x^{2\alpha+1}} e^{e^{-1/\beta x^{2\alpha}}} \left(e^{e^{-1/\beta x^{2\alpha}}} - 1\right)^{\theta_1+\theta_2-1} dx
\end{aligned}
$$

Let $u = e^{e^{-1/\beta x^{2\alpha}}}$, and can solve the integral as

$$
R = \frac{\theta_1}{\theta_1 + \theta_2}, \qquad \theta_1 > 0,\ \theta_2 > 0. \tag{26}
$$

Reliability value is proportion of scale parameter of stress distribution.

### 7.2. Multicomponent Stress-Strength Reliability

[3] first established the reliability of a multicomponent stress-strength model assuming that a common stress $Y$, independent of each $X_i$, can be applied to $l$ identically distinct strength components $X_1, X_2, ..., X_l$. The system with $l$ identical components will functions if $c$ or more number of components operate simultaneously, where $1 \leq$

$c \leq l$. The multicomponent stress-strength model's reliability is determined by

$$
\begin{aligned}
R_{c,l} &= \Pr(\text{at least } c \text{ of the } (X_1, X_2, ..., X_l) \text{ exceeds Y}) \\
&= \sum_{i=c}^{l} \binom{l}{i} \int_0^\infty \big(1 - G(x)\big)^i \big(G(x)\big)^{l-i} dF(x) \\
&= \sum_{i=c}^{l} \binom{l}{i} \int_0^\infty \left(1 - \left(\frac{e^{e^{-1/\beta x^{2\alpha}}} - 1}{e-1}\right)^{\theta_1}\right)^i \left(\frac{e^{e^{-1/\beta x^{2\alpha}}} - 1}{e-1}\right)^{\theta_1(l-i)} \\
&\qquad \frac{2\alpha\theta_2}{\beta(e-1)^{\theta_2}} \frac{e^{-1/\beta x^{2\alpha}}}{x^{2\alpha+1}} e^{e^{-1/\beta x^{2\alpha}}} \left(e^{e^{-1/\beta x^{2\alpha}}} - 1\right)^{\theta_2 - 1} dx \\
&= \frac{2\alpha\theta_2}{\beta(e-1)^{\theta_2+\theta_1 l}} \sum_{i=c}^{l} \binom{l}{i} \int_0^\infty \frac{e^{-1/\beta x^{2\alpha}}}{x^{2\alpha+1}} e^{e^{-1/\beta x^{2\alpha}}} \left(e^{e^{-1/\beta x^{2\alpha}}} - 1\right)^{\theta_1(l-i)+\theta_2-1} \\
&\qquad \left((e-1)^{\theta_1} - \left(e^{e^{-1/\beta x^{2\alpha}}} - 1\right)^{\theta_1}\right)^i dx \\
&= \frac{2\alpha\theta_2}{\beta(e-1)^{\theta_2+\theta_1 l}} \sum_{i=c}^{l}\sum_{j=0}^{i} \binom{l}{i}\binom{i}{j} (-1)^j (e-1)^{\theta_1(i-j)} \\
&\qquad \int_0^\infty \frac{e^{-1/\beta x^{2\alpha}}}{x^{2\alpha+1}} e^{e^{-1/\beta x^{2\alpha}}} \left(e^{e^{-1/\beta x^{2\alpha}}} - 1\right)^{\theta_1(l-i+j)+\theta_2-1} dx \\
&= \sum_{i=c}^{l}\sum_{j=0}^{i} (-1)^j \binom{l}{i}\binom{i}{j} \frac{\theta_2}{\theta_1(l+j-i)+\theta_2}, \qquad \theta_1 > 0,\ \theta_2 > 0.
\end{aligned} \tag{27}
$$

For a parallel system, $c = 1$, then

$$
R_{1,l} = \sum_{i=1}^{l}\sum_{j=0}^{i} (-1)^j \binom{l}{i}\binom{i}{j} \frac{\theta_2}{\theta_1(l+j-i)+\theta_2}, \qquad \theta_1 > 0,\ \theta_2 > 0.
$$

For a series system, $c = l$, then

$$
R_{l,l} = \sum_{j=0}^{l} (-1)^j \binom{l}{j} \frac{\theta_2}{j\theta_1+\theta_2}, \qquad \theta_1 > 0,\ \theta_2 > 0.
$$

### *7.3. Estimation Methods*

In this section, to get the reliability estimate by assessing the parameter estimation, we consider the MLE and MPSE methods on both a single- and multicomponent stress-strength model.

#### *7.3.1. Estimation of Single-component Stress-Strength Model*

*Maximum Likelihood Estimation*

Let $x_1, x_2, ..., x_n$ and $y_1, y_2, ..., y_m$ are independent random variables from PGDUS-PIR$(\alpha, \beta, \theta_1)$ and PGDUS-PIR$(\alpha, \beta, \theta_2)$ respectively. Therefor the joint likelihood function will be

$$L_R = \prod_{i=1}^{n} \frac{2\alpha\theta_1}{\beta(e-1)^{\theta_1}} \frac{e^{-1/\beta x_i^{2\alpha}}}{x_i^{2\alpha+1}} e^{e^{-1/\beta x_i^{2\alpha}}} \big(e^{e^{-1/\beta x_i^{2\alpha}}} - 1\big)^{\theta_1 - 1}$$
$$\prod_{j=1}^{m} \frac{2\alpha\theta_2}{\beta(e-1)^{\theta_2}} \frac{e^{-1/\beta y_j^{2\alpha}}}{y_j^{2\alpha+1}} e^{e^{-1/\beta y_j^{2\alpha}}} \big(e^{e^{-1/\beta y_j^{2\alpha}}} - 1\big)^{\theta_2 - 1}. \tag{28}$$

$$\log L_R = n \log\left(\frac{2\alpha\theta_1}{\beta(e-1)^{\theta_1}}\right) + m \log\left(\frac{2\alpha\theta_2}{\beta(e-1)^{\theta_2}}\right) - \frac{1}{\beta}\big(\sum_{i=1}^{n} x_i^{-2\alpha} + \sum_{j=1}^{m} y_j^{-2\alpha}\big)$$
$$+ \sum_{i=1}^{n} e^{-1/\beta x_i^{2\alpha}} + \sum_{j=1}^{m} e^{-1/\beta y_j^{2\alpha}} + \sum_{i=1}^{n} \log(1/x_i^{2\alpha+1}) + \sum_{j=1}^{m} \log(1/y_j^{2\alpha+1})$$
$$+ (\theta_1 - 1)\sum_{i=1}^{n} \log\big(e^{e^{-1/\beta x_i^{2\alpha}}} - 1\big) + (\theta_2 - 1)\sum_{j=1}^{m} \log\big(e^{e^{-1/\beta y_j^{2\alpha}}} - 1\big).$$

The estimators of the parameters can be obtained by solving the following system of equations by any numerical method respectively. That is,

$$\frac{\partial \log L_R}{\partial \alpha} = 0 =>$$
$$\frac{n+m}{\alpha} + \frac{2}{\beta}\big(\sum_{i=1}^{n} x_i^{-2\alpha}\log(x_i) + \sum_{j=1}^{m} y_j^{-2\alpha}\log(y_j)\big)$$
$$+ \frac{1}{\beta}\big(\sum_{i=1}^{n} x_i^{-2\alpha}\log(x_i) e^{-1/\beta x_i^{2\alpha}} + \sum_{j=1}^{m} y_j^{-2\alpha}\log(y_j) e^{-1/\beta y_j^{2\alpha}}\big)$$
$$- 2\big(\sum_{i=1}^{n}\log(x_i) + \sum_{j=1}^{m}\log(y_j)\big) + \frac{2(\theta_1 - 1)}{\beta}\sum_{i=1}^{n} \frac{e^{-1/\beta x_i^{2\alpha}} e^{e^{-1/\beta x_i^{2\alpha}}}\log(x_i)}{x_i^{-2\alpha}\big(e^{e^{-1/\beta x_i^{2\alpha}}} - 1\big)}$$
$$+ \frac{2(\theta_2 - 1)}{\beta}\sum_{j=1}^{m} \frac{e^{-1/\beta y_j^{2\alpha}} e^{e^{-1/\beta y_j^{2\alpha}}}\log(y_j)}{y_j^{-2\alpha}\big(e^{e^{-1/\beta y_j^{2\alpha}}} - 1\big)} = 0, \tag{29}$$

$$\frac{\partial \log L_R}{\partial \beta} = 0 =>$$
$$-\frac{n+m}{\beta} + \frac{1}{\beta^2}\big(\sum_{i=1}^{n} x_i^{-2\alpha}(e^{-1/\beta x_i^{2\alpha}} + 1) + \sum_{j=1}^{m} y_j^{-2\alpha}(e^{-1/\beta y_j^{2\alpha}} + 1)\big)$$
$$\frac{1}{\beta^2}\left((\theta_1 - 1)\sum_{i=1}^{n} \frac{e^{-1/\beta x_i^{2\alpha}} e^{e^{-1/\beta x_i^{2\alpha}}}}{x_i^{2\alpha}(e^{e^{-1/\beta x_i^{2\alpha}}} - 1)} + (\theta_2 - 1)\sum_{j=1}^{m} \frac{e^{-1/\beta y_j^{2\alpha}} e^{e^{-1/\beta y_j^{2\alpha}}}}{y_j^{2\alpha}(e^{e^{-1/\beta y_j^{2\alpha}}} - 1)}\right) = 0, \tag{30}$$

$$\frac{\partial \log L_R}{\partial \theta_1} = 0 => \frac{n}{\theta_1} - n\log(e-1) + \sum_{i=1}^{n} \log(e^{e^{-1/\beta x_i^{2\alpha}}} - 1) = 0, \tag{31}$$

$$\frac{\partial \log L_R}{\partial \theta_2} = 0 => \frac{m}{\theta_2} - m\log(e-1) + \sum_{j=1}^{m} \log(e^{e^{-1/\beta y_j^{2\alpha}}} - 1) = 0. \tag{32}$$

Suppose, $\hat{\alpha}$ and $\hat{\beta}$ are the MLEs of $\alpha$ and $\beta$ respectively. Then, the MLEs of $\theta_1$ and $\theta_2$ are obtained as follows:

$$\hat{\theta_1} = \frac{n}{n\log(e-1) - \sum_{i=1}^{n} \log(e^{e^{-1/\hat{\beta} x_i^{2\hat{\alpha}}}} - 1)},$$

$$\hat{\theta_2} = \frac{m}{m\log(e-1) - \sum_{j=1}^{m} \log(e^{e^{-1/\hat{\beta} y_j^{2\hat{\alpha}}}} - 1)}.$$

Then, by substituting these values to the reliability (26), we get the estimator of the stress-strength reliability. That is,

$$\hat{R}_{MLE} = \frac{\hat{\theta_1}}{\hat{\theta_1} + \hat{\theta_2}}.$$

*Maximum Product Spacing Estimation*

Let $x_{(1)}, x_{(2)}, ..., x_{(n)}$ represents the order statistics of a random sample of size $n$ obtained from PGDUS-PIR $(\alpha, \beta, \theta_1)$ and let $y_{(1)}, y_{(2)}, ..., y_{(m)}$ be another set of order statistics of a random sample of size $m$ taken from PGDUS-PIR $(\alpha, \beta, \theta_2)$, where both distributions are independent to each other. Then, the uniform spacing of these two independent samples can be defined as follows:

$$S_{1i} = F(x_{(i)}) - F(x_{(i-1)}),$$

$$S_{2j} = G(y_{(j)}) - G(y_{(j-1)}).$$

According to [5], MPS estimate values will maximize the geometric mean of the spacings, which is the difference between the cdf of the consecutive order statistics. So, here the geometric mean for the combined samples can be represented as

$$A = \left( \prod_{i=1}^{n+1} S_{1i} \right)^{1/n+1} \left( \prod_{j=1}^{m+1} S_{2j} \right)^{1/m+1}.$$

To obtain the estimator, we need to maximize the function $A$ with respect to each parameter. For that, take

$$\begin{aligned}\log(A) &= \frac{1}{n+1}\sum_{i=1}^{n}\log(S_{1i}) + \frac{1}{m+1}\sum_{j=1}^{m}\log(S_{2j})\\ &= \frac{1}{n+1}\sum_{i=1}^{n}\log\left(\left(\frac{e^{e^{-1/\beta x_{(i)}^{2\alpha}}}-1}{e-1}\right)^{\theta_1} - \left(\frac{e^{e^{-1/\beta x_{(i-1)}^{2\alpha}}}-1}{e-1}\right)^{\theta_1}\right)\\ &\quad + \frac{1}{m+1}\sum_{j=1}^{m}\log\left(\left(\frac{e^{e^{-1/\beta y_{(j)}^{2\alpha}}}-1}{e-1}\right)^{\theta_2} - \left(\frac{e^{e^{-1/\beta y_{(j-1)}^{2\alpha}}}-1}{e-1}\right)^{\theta_2}\right).\end{aligned}$$

By considering the partial derivative of $\log(A)$ with respect to the unknown parameters $\alpha, \beta, \theta_1$ and $\theta_2$ and equate them with zero.

$$\begin{aligned}\frac{\partial \log A}{\partial \alpha} = 0 => \frac{2\theta_1}{\beta(n+1)}\sum_{i=1}^{n+1}\Bigg(&\frac{x_{(i)}^{-2\alpha}\log(x_{(i)})e^{-1/\beta x_{(i)}^{2\alpha}}e^{e^{-1/\beta x_{(i)}^{2\alpha}}}\left(e^{e^{-1/\beta x_{(i)}^{2\alpha}}}-1\right)^{\theta_1-1}}{\left(e^{e^{-1/\beta x_{(i)}^{2\alpha}}}-1\right)^{\theta_1} - \left(e^{e^{-1/\beta x_{(i-1)}^{2\alpha}}}-1\right)^{\theta_1}}\\ &- \frac{x_{(i-1)}^{-2\alpha}\log(x_{(i-1)})e^{-1/\beta x_{(i-1)}^{2\alpha}}e^{e^{-1/\beta x_{(i-1)}^{2\alpha}}}\left(e^{e^{-1/\beta x_{(i-1)}^{2\alpha}}}-1\right)^{\theta_1-1}}{\left(e^{e^{-1/\beta x_{(i)}^{2\alpha}}}-1\right)^{\theta_1} - \left(e^{e^{-1/\beta x_{(i-1)}^{2\alpha}}}-1\right)^{\theta_1}}\Bigg)\\ + \frac{2\theta_2}{\beta(m+1)}\sum_{j=1}^{m+1}\Bigg(&\frac{y_{(j)}^{-2\alpha}\log(y_{(j)})e^{-1/\beta y_{(j)}^{2\alpha}}e^{e^{-1/\beta y_{(j)}^{2\alpha}}}\left(e^{e^{-1/\beta y_{(j)}^{2\alpha}}}-1\right)^{\theta_2-1}}{\left(e^{e^{-1/\beta y_{(j)}^{2\alpha}}}-1\right)^{\theta_2} - \left(e^{e^{-1/\beta y_{(j-1)}^{2\alpha}}}-1\right)^{\theta_2}}\\ &- \frac{y_{(j-1)}^{-2\alpha}\log(y_{(j-1)})e^{-1/\beta y_{(j-1)}^{2\alpha}}e^{e^{-1/\beta y_{(j-1)}^{2\alpha}}}\left(e^{e^{-1/\beta y_{(j-1)}^{2\alpha}}}-1\right)^{\theta_2-1}}{\left(e^{e^{-1/\beta y_{(j)}^{2\alpha}}}-1\right)^{\theta_2} - \left(e^{e^{-1/\beta y_{(j-1)}^{2\alpha}}}-1\right)^{\theta_2}}\Bigg) = 0.\end{aligned} \tag{33}$$

$$\frac{\partial \log A}{\partial \beta} = 0 =>$$

$$\frac{\theta_1}{\beta^2(n+1)}\sum_{i=1}^{n+1}\Bigg(\frac{x_{(i)}^{-2\alpha}e^{-1/\beta x_{(i)}^{2\alpha}}e^{e^{-1/\beta x_{(i)}^{2\alpha}}}\left(e^{e^{-1/\beta x_{(i)}^{2\alpha}}}-1\right)^{\theta_1-1}}{\left(e^{e^{-1/\beta x_{(i)}^{2\alpha}}}-1\right)^{\theta_1}-\left(e^{e^{-1/\beta x_{(i-1)}^{2\alpha}}}-1\right)^{\theta_1}} - \frac{x_{(i-1)}^{-2\alpha}e^{-1/\beta x_{(i-1)}^{2\alpha}}e^{e^{-1/\beta x_{(i-1)}^{2\alpha}}}\left(e^{e^{-1/\beta x_{(i-1)}^{2\alpha}}}-1\right)^{\theta_1-1}}{\left(e^{e^{-1/\beta x_{(i)}^{2\alpha}}}-1\right)^{\theta_1}-\left(e^{e^{-1/\beta x_{(i-1)}^{2\alpha}}}-1\right)^{\theta_1}}\Bigg)$$
$$+\frac{\theta_2}{\beta^2(m+1)}\sum_{j=1}^{m+1}\Bigg(\frac{y_{(j)}^{-2\alpha}e^{-1/\beta y_{(j)}^{2\alpha}}e^{e^{-1/\beta y_{(j)}^{2\alpha}}}\left(e^{e^{-1/\beta y_{(j)}^{2\alpha}}}-1\right)^{\theta_2-1}}{\left(e^{e^{-1/\beta y_{(j)}^{2\alpha}}}-1\right)^{\theta_2}-\left(e^{e^{-1/\beta y_{(j-1)}^{2\alpha}}}-1\right)^{\theta_2}} - \frac{y_{(j-1)}^{-2\alpha}e^{-1/\beta y_{(j-1)}^{2\alpha}}e^{e^{-1/\beta y_{(j-1)}^{2\alpha}}}\left(e^{e^{-1/\beta y_{(j-1)}^{2\alpha}}}-1\right)^{\theta_2-1}}{\left(e^{e^{-1/\beta y_{(j)}^{2\alpha}}}-1\right)^{\theta_2}-\left(e^{e^{-1/\beta y_{(j-1)}^{2\alpha}}}-1\right)^{\theta_2}}\Bigg)=0. \quad (34)$$

$$\frac{\partial \log A}{\partial \theta_1} = 0 =>$$

$$\frac{1}{n+1}\sum_{i=1}^{n+1}\Bigg(\frac{\left(e^{e^{-1/\beta x_{(i)}^{2\alpha}}}-1\right)^{\theta_1}\log\left(\frac{e^{e^{-1/\beta x_{(i)}^{2\alpha}}}-1}{e-1}\right)}{\left(e^{e^{-1/\beta x_{(i)}^{2\alpha}}}-1\right)^{\theta_1}-\left(e^{e^{-1/\beta x_{(i-1)}^{2\alpha}}}-1\right)^{\theta_1}} - \frac{\left(e^{e^{-1/\beta x_{(i-1)}^{2\alpha}}}-1\right)^{\theta_1}\log\left(\frac{e^{e^{-1/\beta x_{(i-1)}^{2\alpha}}}-1}{e-1}\right)}{\left(e^{e^{-1/\beta x_{(i)}^{2\alpha}}}-1\right)^{\theta_1}-\left(e^{e^{-1/\beta x_{(i-1)}^{2\alpha}}}-1\right)^{\theta_1}}\Bigg)=0. \quad (35)$$

$$\frac{\partial \log A}{\partial \theta_2} = 0 =>$$

$$\frac{1}{m+1}\sum_{j=1}^{m+1}\Bigg(\frac{\left(e^{e^{-1/\beta y_{(j)}^{2\alpha}}}-1\right)^{\theta_2}\log\left(\frac{e^{e^{-1/\beta y_{(j)}^{2\alpha}}}-1}{e-1}\right)}{\left(e^{e^{-1/\beta y_{(j)}^{2\alpha}}}-1\right)^{\theta_2}-\left(e^{e^{-1/\beta y_{(j-1)}^{2\alpha}}}-1\right)^{\theta_2}} - \frac{\left(e^{e^{-1/\beta y_{(j-1)}^{2\alpha}}}-1\right)^{\theta_2}\log\left(\frac{e^{e^{-1/\beta y_{(j-1)}^{2\alpha}}}-1}{e-1}\right)}{\left(e^{e^{-1/\beta y_{(j)}^{2\alpha}}}-1\right)^{\theta_2}-\left(e^{e^{-1/\beta y_{(j-1)}^{2\alpha}}}-1\right)^{\theta_2}}\Bigg)=0. \quad (36)$$

Then, by solving these system of equations using any numerical approach, we can obtain the MPS estimators of each parameter, $\hat{\alpha}_{MPS}$, $\hat{\beta}_{MPS}$, $\hat{\theta}_{1MPS}$, and $\hat{\theta}_{2MPS}$ respectively. Then by replacing this estimator values in (26), MPS estimator of reliability in stress-strength model can be obtain in the form

$$\hat{R}_{MPS} = \frac{\hat{\theta}_{1MPS}}{\hat{\theta}_{1MPS} + \hat{\theta}_{2MPS}}.$$

*7.3.2. Estimation of Multicomponent SSR*

*Maximum Likelihood Estimation*

Let $n$ be the sample size and assume that $X_{i1}, X_{i2}, ..., X_{il}$ and $Y_i,\ i = 1, 2, ..., n$ be the observed data from PGDUS-PIR$(\alpha, \beta, \theta_1)$ and PGDUS-PIR$(\alpha, \beta, \theta_2)$ respectively. Hence, the likelihood function of these unknown parameters can be written as

$$\begin{aligned} L_{c,l} &= \prod_{i=1}^{n} \Big( \prod_{j=c}^{l} f(x_{ij})\Big) g(y_i) = \prod_{i=1}^{n} \left( \prod_{j=c}^{l} \left( \frac{2\alpha\theta_1}{\beta(e-1)^{\theta_1}} \frac{e^{-1/\beta x_{ij}^{2\alpha}}}{x_{ij}^{2\alpha+1}} e^{e^{-1/\beta x_{ij}^{2\alpha}}} \left(e^{e^{-1/\beta x_{ij}^{2\alpha}}} - 1\right)^{\theta_1 - 1} \right) \right. \\ &\qquad \left. \frac{2\alpha\theta_2}{\beta(e-1)^{\theta_2}} \frac{e^{-1/\beta y_i^{2\alpha}}}{y_i^{2\alpha+1}} e^{e^{-1/\beta y_i^{2\alpha}}} \left(e^{e^{-1/\beta y_i^{2\alpha}}} - 1\right)^{\theta_2 - 1} \right) \\ &= \left(\frac{2\alpha}{\beta}\right)^{n(l+1)} \frac{\theta_1^{nl}}{(e-1)^{nl\theta_1 + n\theta_2}} \prod_{i=1}^{n} \left( \prod_{j=c}^{l} \left( \frac{e^{-1/\beta x_{ij}^{2\alpha}}}{x_{ij}^{2\alpha+1}} e^{e^{-1/\beta x_{ij}^{2\alpha}}} \left(e^{e^{-1/\beta x_{ij}^{2\alpha}}} - 1\right)^{\theta_1 - 1} \right) \right) \\ &\qquad \left. \frac{e^{-1/\beta y_i^{2\alpha}}}{y_i^{2\alpha+1}} e^{e^{-1/\beta y_i^{2\alpha}}} \left(e^{e^{-1/\beta y_i^{2\alpha}}} - 1\right)^{\theta_2 - 1} \right). \end{aligned} \tag{37}$$

$$\begin{aligned} \log L_{c,l} &= n(l+1)\big(\log 2 + \log(\alpha) - \log(\beta)\big) + nl\log(\theta_1) - n(l\theta_1 + \theta_2)\log(e-1) \\ &\quad - (2\alpha+1)\Big(\sum_{i=1}^{n}\sum_{j=1}^{m}\log(x_{ij}) + \sum_{i=1}^{n}\log(y_j)\Big) - \frac{1}{\beta}\Big(\sum_{i=1}^{n}\sum_{j=1}^{m} x_{ij}^{-2\alpha} + \sum_{i=1}^{n} y_i^{-2\alpha}\Big) \\ &\quad + \sum_{i=1}^{n}\sum_{j=1}^{m} e^{-1/\beta x_{ij}^{2\alpha}} + \sum_{i=1}^{n} e^{-1/\beta y_i^{2\alpha}} + (\theta_1 - 1)\sum_{i=1}^{n}\sum_{j=1}^{m}\log\left(e^{e^{-1/\beta x_{ij}^{2\alpha}}} - 1\right) \\ &\quad + (\theta_2 - 1)\sum_{i=1}^{n}\log\left(e^{e^{-1/\beta y_i^{2\alpha}}} - 1\right). \end{aligned}$$

Maximize the function $\log L_{c,l}$ with respect to each unknown parameters to obtain its estimators. For that, consider the partial derivative of $\log L_{c,l}$ with respect to $\alpha,\ \beta,\ \theta_1$

and $\theta_2$ and equate them with zero. That is,

$$\frac{\partial \log L_{c,l}}{\partial \alpha} = 0 => \\ \frac{n(l+1)}{\alpha} - 2\Big(\sum_{i=1}^{n}\sum_{j=1}^{m}\log(x_{ij}) + \sum_{i=1}^{n}\log(y_j)\Big) + \frac{2}{\beta}\Bigg(\sum_{i=1}^{n}\sum_{j=1}^{m}\frac{\log(x_{ij})}{x_{ij}^{2\alpha}} + \sum_{i=1}^{n}\frac{\log(y_j)}{y_i^{2\alpha}}\Bigg) \\ - \frac{1}{\beta}\Bigg(\sum_{i=1}^{n}\sum_{j=1}^{m}\frac{e^{-1/\beta x_{ij}^{2\alpha}}\log(x_{ij})}{x_{ij}^{2\alpha}} + \sum_{i=1}^{n}\frac{e^{-1/\beta y_i^{2\alpha}}\log(y_j)}{y_i^{2\alpha}}\Bigg) \\ + \frac{2(\theta_1-1)}{\beta}\sum_{i=1}^{n}\sum_{j=1}^{m}\frac{\log(x_{ij})e^{-1/\beta x_{ij}^{2\alpha}}e^{e^{-1/\beta x_{ij}^{2\alpha}}}}{x_{ij}^{2\alpha}(e^{e^{-1/\beta x_{ij}^{2\alpha}}}-1)} + \frac{2(\theta_2-1)}{\beta}\sum_{i=1}^{n}\frac{\log(y_i)e^{-1/\beta y_i^{2\alpha}}e^{e^{-1/\beta y_i^{2\alpha}}}}{y_i^{2\alpha}(e^{e^{-1/\beta y_i^{2\alpha}}}-1)} = 0 \tag{38}$$

$$\frac{\partial \log L_{c,l}}{\partial \beta} = 0 => \\ \frac{-n(l+1)}{\beta} + \frac{1}{\beta^2}\Bigg(\sum_{i=1}^{n}\sum_{j=1}^{m}\frac{(e^{e^{-1/\beta x_{ij}^{2\alpha}}}+1)}{x_{ij}^{2\alpha}} + \sum_{i=1}^{n}\frac{(e^{e^{-1/\beta y_i^{2\alpha}}}+1)}{y_i^{2\alpha}} \\ + (\theta_1-1)\sum_{i=1}^{n}\sum_{j=1}^{m}\frac{e^{-1/\beta x_{ij}^{2\alpha}}e^{e^{-1/\beta x_{ij}^{2\alpha}}}}{x_{ij}^{2\alpha}(e^{e^{-1/\beta x_{ij}^{2\alpha}}}-1)} + (\theta_2-1)\sum_{i=1}^{n}\frac{e^{-1/\beta y_i^{2\alpha}}e^{e^{-1/\beta y_i^{2\alpha}}}}{y_i^{2\alpha}(e^{e^{-1/\beta y_i^{2\alpha}}}-1)}\Bigg) = 0 \tag{39}$$

$$\frac{\partial \log L_{c,l}}{\partial \theta_1} = 0 => \frac{nl}{\theta_1} - nl\log(e-1) + \sum_{i=1}^{n}\sum_{j=1}^{m}\log\left(e^{e^{-1/\beta x_{ij}^{2\alpha}}}-1\right) = 0 \tag{40}$$

$$\frac{\partial \log L_{c,l}}{\partial \theta_2} = 0 => \frac{n}{\theta_2} - n\log(e-1) + \sum_{i=1}^{n}\log\left(e^{e^{-1/\beta y_i^{2\alpha}}}-1\right) = 0 \tag{41}$$

By solving above equations, we can obtain the parameter estimators respectively. Suppose, $\hat{\alpha}$ and $\hat{\beta}$ are the MLEs of $\alpha$ and $\beta$, then we can obtain $\hat{\theta}_1$ and $\hat{\theta}_2$ as follows:

$$\hat{\theta}_1 = \frac{nl}{nl\log(e-1) - \sum_{i=1}^{n}\sum_{j=1}^{m}\log\left(e^{e^{-1/\beta x_{ij}^{2\alpha}}}-1\right)},$$
$$\hat{\theta}_2 = \frac{n}{n\log(e-1) - \sum_{i=1}^{n}\log\left(e^{e^{-1/\beta y_i^{2\alpha}}}-1\right)}.$$

Then, by substituting these estimator to the reliability (27), obtain the reliability estimator in multi component stress-strength model. That is,

$$\hat{R}_{c,l-MLE} = \sum_{i=c}^{l}\sum_{j=0}^{i}(-1)^j\binom{l}{i}\binom{i}{j}\frac{\hat{\theta}_2}{\hat{\theta}_1(l+j-i)+\hat{\theta}_2}.$$

*Maximum Product Spacing Estimation*

Let $x_{(1)}, x_{(2)}, ..., x_{(nl)}$ and $y_{(1)}, y_{(2)}, ..., y_{(n)}$ are the ordered observed data, taken from PGDUS-PIR $(\alpha, \beta, \theta_1)$ and PGDUS-PIR $(\alpha, \beta, \theta_2)$ independent distributions with cdf $F(x)$ and $G(y)$ respectively. MPS estimators are the parameter values that maximizes the geometric mean of the uniform spacings of random samples. Let

$$S_{1j} = F(x_{(j)}) - F(x_{(j-1)}), \quad j = 1, 2, ..., n,$$

and

$$S_{2i} = G(y_{(i)}) - G(y_{(i-1)}), \quad i = 1, 2, ..., n$$

are the uniform spacing of the sample from each distributions. Then, the geometric mean of the combined sample will be the form

$$A_M = \left( \prod_{j=1}^{nl+1} S_{1j} \right)^{1/nl+1} \left( \prod_{i=1}^{n+1} S_{2i} \right)^{1/n+1}.$$

By taking its partial derivative with respect to $\theta_1$ and $\theta_2$ and equating it with zero, and solving by any numerical method, we get MPS estimators. That is,

$$\frac{\partial \log A_M}{\partial \theta_1} = 0 =>$$

$$\frac{1}{nl+1} \sum_{j=1}^{nl+1} \left( \frac{\left(e^{e^{-1/\beta x_{(j)}^{2\alpha}}} - 1\right)^{\theta_1} \log\left(\frac{e^{e^{-1/\beta x_{(j)}^{2\alpha}}} - 1}{e-1}\right)}{\left(e^{e^{-1/\beta x_{(j)}^{2\alpha}}} - 1\right)^{\theta_1} - \left(e^{e^{-1/\beta x_{(j-1)}^{2\alpha}}} - 1\right)^{\theta_1}} - \frac{\left(e^{e^{-1/\beta x_{(j-1)}^{2\alpha}}} - 1\right)^{\theta_1} \log\left(\frac{e^{e^{-1/\beta x_{(j-1)}^{2\alpha}}} - 1}{e-1}\right)}{\left(e^{e^{-1/\beta x_{(j)}^{2\alpha}}} - 1\right)^{\theta_1} - \left(e^{e^{-1/\beta x_{(j-1)}^{2\alpha}}} - 1\right)^{\theta_1}} \right) = 0. \quad (42)$$

$$\frac{\partial \log A_M}{\partial \theta_2} = 0 =>$$

$$\frac{1}{n+1} \sum_{i=1}^{n+1} \left( \frac{\left(e^{e^{-1/\beta y_{(i)}^{2\alpha}}} - 1\right)^{\theta_2} \log\left(\frac{e^{e^{-1/\beta y_{(i)}^{2\alpha}}} - 1}{e-1}\right)}{\left(e^{e^{-1/\beta y_{(i)}^{2\alpha}}} - 1\right)^{\theta_2} - \left(e^{e^{-1/\beta y_{(i-1)}^{2\alpha}}} - 1\right)^{\theta_2}} - \frac{\left(e^{e^{-1/\beta y_{(i-1)}^{2\alpha}}} - 1\right)^{\theta_2} \log\left(\frac{e^{e^{-1/\beta y_{(i-1)}^{2\alpha}}} - 1}{e-1}\right)}{\left(e^{e^{-1/\beta y_{(i)}^{2\alpha}}} - 1\right)^{\theta_2} - \left(e^{e^{-1/\beta y_{(i-1)}^{2\alpha}}} - 1\right)^{\theta_2}} \right) = 0. \quad (43)$$

Then by substituting the parameter estimates $\hat{\theta}_{1M-PS}$ and $\hat{\theta}_{2M-PS}$ in (27), we get the reliability estimator of the model. That is,

$$\hat{R}_{c,l-MPS} = \sum_{i=c}^{l}\sum_{j=0}^{i}(-1)^j \binom{l}{i}\binom{i}{j} \frac{\hat{\theta}_{2M-PS}}{\hat{\theta}_{1M-PS}(l+j-i)+\hat{\theta}_{2M-PS}}.$$

### *7.4. Simulation Study*

Simulation study for single-component and multi-component stress-strength models are carried out to examine the reliability performance with respect to their biases and MSE values. Here, we consider both MLE and MPSE methods in each model.

**Table 6.** Simulation study for Stress-Strength reliability for $\alpha = 0.3, \beta = 0.8, \theta_1 = 0.07, \theta_2 = 0.09, R = 0.4375$

| method | (n,m) | $\hat{R}$ | Bias | MSE |
|---|---|---|---|---|
| MLE | (10,10) | 0.54772 | -0.43202 | 0.18962 |
| | (25,25) | 0.55938 | -0.42352 | 0.18699 |
| | (50,50) | 0.52957 | -0.41102 | 0.18226 |
| | (100,100) | 0.53028 | -0.38447 | 0.17313 |
| | (200,200) | 0.53028 | -0.33144 | 0.15485 |
| MPS | (10,15) | 0.55741 | 0.08684 | 0.04449 |
| | (30,25) | 0.52377 | 0.08156 | 0.01762 |
| | (50,50) | 0.48978 | 0.05107 | 0.00861 |
| | (80,90) | 0.47942 | 0.04106 | 0.00543 |
| | (100,100) | 0.45481 | 0.01845 | 0.00296 |
| | (200,150) | 0.43057 | -0.00492 | 0.00145 |

**Table 7.** Simulation study for Multicomponent Stress-Strength reliability for $\alpha = 0.3, \beta = 0.8, \theta_1 = 0.7, \theta_2 = 0.9$, $R_{3,6} = 0.49574$, $R_{2,8} = 0.72785$

| Method | (c,l) | n | $\hat{R}_{cl}$ | Bias | MSE |
|---|---|---|---|---|---|
| MLE | (3,6) | 10 | 0.28401 | -0.21173 | 0.04483 |
| | | 30 | 0.43965 | -0.05609 | 0.00315 |
| | | 80 | 0.48507 | -0.01067 | 0.00012 |
| | | 100 | 0.49062 | -0.00511 | 2.615e-05 |
| | | 150 | 0.49461 | -0.00112 | 1.262e-06 |
| | (2,8) | 10 | 0.66154 | -0.06631 | 0.00439 |
| | | 30 | 0.67101 | -0.05684 | 0.00323 |
| | | 80 | 0.74944 | 0.02159 | 0.00047 |
| | | 100 | 0.73936 | 0.01151 | 0.00013 |
| | | 150 | 0.72481 | -0.00304 | 9.2655e-06 |
| MPS | (3,6) | 10 | 0.20951 | -0.28622 | 0.081924 |
| | | 30 | 0.38445 | -0.11129 | 0.01238 |
| | | 80 | 0.46522 | -0.03052 | 0.00093 |
| | | 100 | 0.51151 | 0.01578 | 0.00025 |
| | | 150 | 0.49246 | -0.00328 | 1.073e-05 |
| | (2,8) | 10 | 0.55577 | -0.17208 | 0.02961 |
| | | 30 | 0.66128 | -0.06657 | 0.00443 |
| | | 80 | 0.75970 | 0.03185 | 0.00102 |
| | | 100 | 0.73889 | 0.01104 | 0.00012 |

Table 6 shows the simulation results for a single component stress-strength model with initial parameter values as $(0.3, 0.8, 0.07, 0.09)$ respectively. The calculated value of the reliability is 0.4375. Here, we are considering different sets of the sample size in each estimation method and it is shows that the bias and MSE values of the reliability measure decrease with increasing sample size.

Table 7 gives the simulation results of a multicomponent stress-strength model for MLE and MPSE methods by considering two sets of $(c, l) = \{(3, 6), (2, 8)\}$ and initial parameter values $(0.3, 0.8, 0.7, 0.9)$ respectively. The calculated reliability measures are $R_{3,6} = 0.49574$ and $R_{2,8} = 0.72785$. In both MLE and MPSE methods, for each set of $(c, l)$, the bias and MSE values of the reliability measure reduce with larger samples.

## 8. Data Analysis

Consider the computation of $Pr(Y < X)$ for the carbon fiber data given in [2]. We used Kolmogorov-Smirnov(K-S) statistic and CVM statistic with p-values to check the fit of the model.

Here, we have two data that reflect the GPA strength of single carbon fibers at two different lengths. In this analysis work, $Y$ represents the GPA strength of the carbon fiber with length 10mm (Table 8) and $X$ represents the GPA strength of the carbon fiber with length 20mm (Table 9) respectively.

**Table 8.** Carbon Fibers Data with length of 10mm

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 1.901 | 2.132 | 2.203 | 2.228 | 2.257 | 2.350 | 2.361 | 2.396 | 3.886 |
| 2.397 | 2.445 | 2.454 | 2.474 | 2.518 | 2.522 | 2.525 | 2.532 | 3.971 |
| 2.575 | 2.614 | 2.616 | 2.618 | 2.624 | 2.659 | 2.675 | 2.738 | 4.024 |
| 2.740 | 2.856 | 2.917 | 2.928 | 2.937 | 2.937 | 2.977 | 2.996 | 4.027 |
| 3.030 | 3.125 | 3.139 | 3.145 | 3.220 | 3.223 | 3.235 | 3.243 | 4.225 |
| 3.264 | 3.272 | 3.294 | 3.332 | 3.346 | 3.377 | 3.408 | 3.435 | 4.395 |
| 3.493 | 3.501 | 3.537 | 3.554 | 3.562 | 3.628 | 3.852 | 3.871 | 5.020 |

**Table 9.** Carbon Fibers Data with length of 20mm

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 1.312 | 1.314 | 1.479 | 1.552 | 1.700 | 1.803 | 1.861 | 1.865 | 1.944 |
| 1.958 | 1.966 | 1.997 | 2.006 | 2.021 | 2.027 | 2.055 | 2.063 | 2.098 |
| 2.14 | 2.179 | 2.224 | 2.240 | 2.253 | 2.270 | 2.272 | 2.274 | 2.301 |
| 2.301 | 2.359 | 2.382 | 2.382 | 2.426 | 2.434 | 2.435 | 2.478 | 2.490 |
| 2.511 | 2.514 | 2.535 | 2.554 | 2.566 | 2.57 | 2.586 | 2.629 | 2.633 |
| 2.642 | 2.648 | 2.684 | 2.697 | 2.726 | 2.770 | 2.773 | 2.800 | 2.809 |
| 2.818 | 2.821 | 2.848 | 2.88 | 2.954 | 3.012 | 3.067 | 3.084 | 3.090 |
| 3.096 | 3.128 | 3.233 | 3.433 | 3.585 | 3.585 | | | |

Here, we used MLE and MPS estimation techniques to compute the estimate of $Pr(Y < X)$.

**Table 10.** Data Analysis

| Method | $\hat{R}$ | | K-S (p-value) | CVM (p-value) |
|---|---|---|---|---|
| MLE | 0.40491 | X | 0.096102 (0.6057) | 0.087031 (0.6538) |
| | | Y | 0.12654 (0.2191) | 0.34922 (0.0986) |
| MPSE | 0.46612 | X | 0.1042 (0.5008) | 0.11219 (0.5296) |
| | | Y | 0.14846 (0.09551) | 0.4724 (0.0501) |

From Table 10, we can observe that the carbon fiber data fit the PGDUS-PIR distribution, since the p-value $\geq$ 0.05 and we obtained the estimate values of the $Pr(Y < X)$ model using both MLE and MPSE methods.

## 9. Summary

In this work, we applied the PGDUS transformation to the PIR distribution to create a new lifetime distribution called PGDUS-PIR with parameters $\alpha, \beta$, and $\theta$. This distribution is applicable to the reliability analysis of the parallel system. Basic statistical properties such as moments, quantile functions, order statistics, and entropy are derived. Figure 1, Figure 2 and Figure 3 show the pdf, cdf, and failure rate plot of the PGDUS-PIR distribution for different values of parameters. To estimate the unknown parameters $\alpha, \beta$, and $\theta$, the maximum likelihood and maximum product spacing estimation methods were used. The simulation studies' findings (Table 1) demonstrate that for both the MLE and MPSE approaches, parameter values, since biases and the MSE decrease with increasing sample size. To show the effectiveness and applicability of the proposed distribution, it was fitted to two real-life datasets, one is a set of successive values of precipitation on March for Minneapolis/St.Paul and another that documents the daily cases of COVID-19 across the world reported by WHO(2020). Compared to alternative distributions such as PGDUS-inverse Kumaraswamy, DUS-powered inverse Rayleigh, and the powered inverse Rayleigh distribution, the proposed distribution, PGDUS-powered inverse Rayleigh distribution provided the best fit (see Table 3 and Table 5). The empirical cdf of the successive values of March precipitation for Minneapolis/St.Paul and the theoretical cdf of PGDUS-PIR, DUS-PIR, and PIR are shown in Figure 4. Similarly, the empirical cdf and the theoretical cdf of PGDUS-PIR, PGDUS-IK, DUS-PIR and PIR for daily cases of COVID-19 data are shown in Figure 5.

The concept of single- and multicomponent stress-strength reliability is discussed. We assumed that the stress variable and the strength variable are distributed as independent PGDUS-PIR distributions. We found that reliability could be estimated in both single- and multicomponent stress-strength models using the MLE and MPSE approaches. Based on simulation studies conducted for both models (Table 6 and Table 7), it can be observed that the estimated reliability values are close to their distributional values and the biases and MSEs decrease with increasing sample size. The computation of $Pr(Y < X)$ in practical situations was demonstrated by the GPA strength of single carbon fibers with lengths of 10 mm and 20 mm. The goodness of fit of the estimators for each data set was assessed using K-S statistics and CVM statistics with p-values, and the outcomes were adequate and satisfactory.

Thus, the PGDUS-PIR distribution turns out to be a more accurate model for the data in hand than the PGDUS-IK, DUS-PIR, and PIR distributions. Further data analysis with censored observations, PH model analysis, etc still needs to be addressed. That is left for future work.

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